

A $PSL_2(\mathbb{C})$ Casson invariant

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Abstract. We use intersection theory techniques to define an invariant of closed 3-manifolds counting the characters of irreducible representations of the fundamental group in $PSL_2(\mathbb{C})$. We note several properties of the invariant and compute the invariant for certain Seifert fibered spaces and for some Dehn surgeries on twist knots. We discuss the relationship between this invariant and the $SL_2(\mathbb{C})$ -invariant defined in [6].

1 Introduction

The purpose of this paper is to use intersection theory techniques to define an invariant of closed 3-manifolds which will count the equivalence classes of irreducible representations of the fundamental group in $PSL_2(\mathbb{C})$. The invariant defined here may be viewed as a $PSL_2(\mathbb{C})$ analog of the Casson invariant.

Recall that the Casson invariant counts equivalence classes of irreducible representations of the fundamental group in $SU(2)$. This invariant has been generalized to count representations in a number of other groups. Of particular interest for us are the generalizations to $SO(3)$ found in [8] and to $SL_2(\mathbb{C})$ found in [6]. The $SO(3)$ invariant, like the Casson invariant, has a surgery formula for rational homology spheres, as shown in [8]. A surgery formula for most surgeries on knots in homology spheres for the $SL_2(\mathbb{C})$ -invariant can be found in [6], as corrected in [7], and further calculations for the $SL_2(\mathbb{C})$ -invariant can be found in [2].

Recall that $SO(3)$ is isomorphic to $PSU(2) = SU(2)/\pm I$. In [8], this fact is used to relate the piece of the $SO(3)$ -invariant which counts representations which lift to $SU(2)$ to the $SU(2)$ -invariant, and eventually to compute the $SO(3)$ -invariant. More specifically, we define the $SO(3)$ -invariant of a closed 3-manifold Σ to be a sum of invariants $\lambda_\alpha(\Sigma)$, where $\alpha \in H^2(\Sigma; \mathbb{Z}_2)$ and $\lambda_\alpha(\Sigma)$ counts the equivalence classes of representations ρ with $w_2(\rho) = \alpha$. It is easy to check that $\lambda_0(\Sigma) = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|} \lambda_{SU(2)}(\Sigma)$. In fact, in the end it is shown that $\lambda_\alpha(\Sigma) = \lambda_0(\Sigma)$

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for every $\alpha \in H^2(\Sigma; \mathbb{Z}_2)$. Since $H^1(\Sigma; \mathbb{Z}_2)$ and $H^2(\Sigma; \mathbb{Z}_2)$ are isomorphic, it follows that the $SO(3)$ -invariant is equal to the $SU(2)$ -invariant.

Here, we will relate the portion of the $PSL_2(\mathbb{C})$ -invariant which counts representations which lift to $SL_2(\mathbb{C})$ to the invariant of [6] in a similar fashion. Again, we may define invariants $\lambda_\alpha(\Sigma)$ counting the equivalence classes of irreducible $PSL_2(\mathbb{C})$ -representations ρ with $w_2(\rho) = \alpha$. As in the case of $SO(3)$, we will see that $\lambda_0(\Sigma) = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|} \lambda_{SL_2(\mathbb{C})}(\Sigma)$. Interestingly, however, in contrast to the $SO(3)$ -theory, we will show that in general $\lambda_\alpha(\Sigma) \neq \lambda_0(\Sigma)$, so the $PSL_2(\mathbb{C})$ -invariant does *not* equal the $SL_2(\mathbb{C})$ -invariant.

In Section 2, we define the invariant $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$, and in Section 3 we establish several of its properties. In Section 4 we compute the $PSL_2(\mathbb{C})$ - and $SL_2(\mathbb{C})$ -invariants for certain Seifert fibered spaces and for certain surgeries on twist knots, and in Section 5 we state several observations, questions, and conjectures concerning the invariant and its relationship to the $SL_2(\mathbb{C})$ -invariant.

2 Definition of the $PSL_2(\mathbb{C})$ -invariant

Given a finitely generated group π , denote by $R(\pi)$ the space of representations $\rho: \pi \rightarrow PSL_2(\mathbb{C})$. It is well known that $R(\pi)$ has the structure of a complex affine algebraic set. (See [3], for example.) Let $X(\pi)$ denote the algebro-geometric quotient under the natural action of $PSL_2(\mathbb{C})$ on $R(\pi)$. Note that there is a surjective quotient map $t: R(\pi) \rightarrow X(\pi)$. By analogy with the $SL_2(\mathbb{C})$ -theory, we call the elements of $X(\pi)$ the $PSL_2(\mathbb{C})$ -characters of π .

A representation $\rho \in R(\pi)$ is *reducible* if it is conjugate to a representation whose image is upper triangular; all other representations are *irreducible*. Let $R^*(\pi)$ denote the subspace of $R(\pi)$ consisting of irreducible representations. Let $X^*(\pi)$ be the subspace of $X(\pi)$ of characters of irreducible representations. Given a manifold Σ , we denote by $R(\Sigma)$ the variety of $PSL_2(\mathbb{C})$ representations of $\pi_1 \Sigma$ and by $X(\Sigma)$ the associated character variety.

Suppose now Σ is a closed, orientable 3-manifold with a Heegaard splitting (W_1, W_2, F) . Here, F is a closed orientable surface embedded in Σ , and W_1 and W_2 are handlebodies with boundaries $\partial W_1 = F = \partial W_2$ such that $\Sigma = W_1 \cup_F W_2$. The inclusion maps $F \hookrightarrow W_i$ and $W_i \hookrightarrow \Sigma$ induce a pushout diagram

$$\begin{array}{ccc} & \pi_1 W_1 & \\ \nearrow & & \searrow \\ \pi_1 F & & \pi_1 \Sigma \\ \searrow & & \nearrow \\ & \pi_1 W_2 & \end{array}$$

This diagram induces a diagram of the associated representation varieties, where all arrows are reversed and are injective rather than surjective. On the level of

character varieties, this gives a pullback diagram

$$\begin{array}{ccc} & X(W_1) & \\ \swarrow & & \searrow \\ X(F) & & X(\Sigma) \\ \swarrow & & \searrow \\ & X(W_2) & \end{array}$$

This identifies $X(\Sigma)$ as the intersection

$$X(\Sigma) = X(W_1) \cap X(W_2) \subset X(F).$$

To generalize the Casson invariant to $PSL_2(\mathbb{C})$, we should define an intersection number for $X(W_1)$ and $X(W_2)$ in $X(F)$. However the spaces involved are not manifolds, so we must do this carefully. To begin with, the spaces are stratified according to the reducibility of the characters. Since we are primarily interested in counting irreducible characters, we may avoid this problem by defining an intersection number which counts only those characters which lie in compact subvarieties of $X^*(F)$. This is completely analogous to the approach taken in [6] to define an invariant counting irreducible $SL_2(\mathbb{C})$ characters.

However, unlike the $SL_2(\mathbb{C})$ theory the variety $X^*(F)$ is not a manifold, but rather an orbifold. Therefore we must define an orbifold intersection number, counting points in the intersection with weight $\pm 1/n$, where n is the order of the stabilizer group. This is completely analogous to the approach taken in [8] to define an invariant counting equivalence classes of irreducible $SO(3)$ -representations.

We now describe the orbifold structure on $X^*(F)$. Let $D \subset PSL_2(\mathbb{C})$ denote the subgroup consisting of the images of the diagonal matrices and the matrices $\pm \begin{pmatrix} 0 & z \\ -z^{-1} & 0 \end{pmatrix}$ where $z \in \mathbb{C}$, $z \neq 0$. Let Z denote the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ -subgroup of D . For an irreducible representation ρ which is conjugate to a representation into $D - Z$, it is easy to check that $PSL_2(\mathbb{C})$ does not act freely on the orbit $O(\rho)$, but rather acts with \mathbb{Z}_2 -isotropy. Similarly, if ρ is conjugate to a representation onto Z , then $PSL_2(\mathbb{C})$ acts with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ -isotropy. However, $PSL_2(\mathbb{C})$ does act freely on the orbits of all other irreducible representations.

There are natural orientations on all the character varieties determined by their complex structures. The invariant $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$ is defined as an oriented orbifold intersection number of $X^*(W_1)$ and $X^*(W_2)$ in $X^*(F)$ which counts only compact, zero-dimensional components of the intersection.

Specifically, there exist a compact neighborhood U of the zero-dimensional components of $X^*(W_1) \cap X^*(W_2)$ which is disjoint from the higher dimensional components of the intersection and an isotopy $h: X^*(F) \rightarrow X^*(F)$ supported in U such that $h(X^*(W_1))$ and $X^*(W_2)$ intersect transversely in U . This isotopy may not preserve the lower strata consisting of characters of representations into D and into Z . To account for this, we count a character χ in $h(X^*(W_1)) \cap X^*(W_2)$ with weight $\beta_\chi = 1$ if χ is not the character of a representation that conjugates into D , with weight $\beta_\chi = 1/2$ if χ is the character of a representation which conjugates into D but not into Z , and with weight $\beta_\chi = 1/4$ if χ is the character of a representation which conjugates into Z . Now set $\varepsilon_\chi = \pm \beta_\chi$, depending on whether the orientation of

$h(X^*(W_1))$ followed by that of $X^*(W_2)$ agrees with or disagrees with the orientation of $X^*(F)$ at χ .

Definition 2.1 Let $\lambda_{PSL_2(\mathbb{C})}(\Sigma) = \sum_{\chi} \varepsilon_{\chi}$, where the sum is over all zero-dimensional components χ of the intersection $h(X^*(W_1)) \cap X^*(W_2)$.

Standard arguments show that the quantity $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$ is independent of the choice of isotopy used to define it and of the choice of Heegaard splitting of Σ . (See [6] and [8], for example.) Thus we obtain

Theorem 2.2 $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$ is a well-defined invariant of Σ .

Also as in the $SL_2(\mathbb{C})$ -theory, this definition agrees with the algebro-geometric intersection number. Thus, for χ a zero-dimensional component of the intersection $X^*(W_1) \cap X^*(W_2) \subset X^*(F)$, if we define n_{χ} to be the coefficient of χ in the intersection cycle $X^*(W_1) \cdot X^*(W_2)$, we obtain

Theorem 2.3 $\lambda_{PSL_2(\mathbb{C})}(\Sigma) = \sum_{\chi} n_{\chi}$, where the sum is taken over all zero-dimensional components of $X^*(W_1) \cap X^*(W_2)$.

See [6] for details.

3 Properties of the invariant

We now establish some basic properties of the $PSL_2(\mathbb{C})$ Casson invariant. These follow quite easily from the definition and the relationship between $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$. We show

Theorem 3.1 The invariant $\lambda_{PSL_2(\mathbb{C})}$ has the following properties:

- (i) For any 3-manifold Σ , $\lambda_{PSL_2(\mathbb{C})}(\Sigma) \geq 0$. If $X^*(W_1) \cap X^*(W_2)$ has at least one zero-dimensional component, then $\lambda_{PSL_2(\mathbb{C})}(\Sigma) > 0$.
- (ii) If Σ is hyperbolic, then $\lambda_{PSL_2(\mathbb{C})}(\Sigma) > 0$.
- (iii) If $\lambda_{PSL_2(\mathbb{C})}(\Sigma) > 0$, then there exists an irreducible representation $\rho: \pi_1 \Sigma \rightarrow PSL_2(\mathbb{C})$. In particular, if $\pi_1 \Sigma$ is cyclic, then $\lambda_{PSL_2(\mathbb{C})}(\Sigma) = 0$.
- (iv) If Σ is a \mathbb{Z}_2 homology 3-sphere, then $\lambda_{PSL_2(\mathbb{C})}(\Sigma) = \lambda_{SL_2(\mathbb{C})}(\Sigma)$.
- (v) $\lambda_{PSL_2(\mathbb{C})}(-\Sigma) = \lambda_{PSL_2(\mathbb{C})}(\Sigma)$, where $-\Sigma$ is Σ with the opposite orientation.
- (vi) For rational homology spheres Σ_1 and Σ_2 , we have

$$\lambda_{PSL_2(\mathbb{C})}(\Sigma_1 \# \Sigma_2) = \lambda_{PSL_2(\mathbb{C})}(\Sigma_1) + \lambda_{PSL_2(\mathbb{C})}(\Sigma_2).$$

Proof (i) This is an immediate consequence of Definition 2.1 and Theorem 2.3.

- (ii) The hyperbolic structure on Σ is a discrete, faithful, irreducible representation $\pi_1(\Sigma) \rightarrow PSL_2(\mathbb{C})$. By the main theorem of [10], the discrete, faithful representations form an open subset of $X(\Sigma)$. On the other hand, by Mostow rigidity, the discrete faithful representation in $PSL_2(\mathbb{C})$ is unique. It follows that the character of the representation is isolated in $X(\Sigma)$, and thus $X^*(W_1) \cap X^*(W_2)$ contains a zero-dimensional component. Hence (ii) follows from (i).
- (iii) This is an immediate consequence of Definition 2.1.
- (iv) The quotient map $SL_2(\mathbb{C}) \rightarrow PSL_2(\mathbb{C})$ induces orientation-preserving maps $X_{SL_2(\mathbb{C})}(\Sigma) \rightarrow X(\Sigma)$, $X_{SL_2(\mathbb{C})}(W_i) \rightarrow X(W_i)$, and $X_{SL_2(\mathbb{C})}(F) \rightarrow X(F)$,

where $X_{SL_2(\mathbb{C})}$ denotes the $SL_2(\mathbb{C})$ -character variety. Moreover the isotopy $h: X^*(F) \rightarrow X^*(F)$ used to make $X(W_1)$ transverse to $X(W_2)$ induces an isotopy $h: X_{SL_2(\mathbb{C})}^*(F) \rightarrow X_{SL_2(\mathbb{C})}^*(F)$ making $X_{SL_2(\mathbb{C})}(W_1)$ transverse to $X_{SL_2(\mathbb{C})}(W_2)$. If Σ is a \mathbb{Z}_2 -homology sphere, then the orientation-preserving map $X_{SL_2(\mathbb{C})}(\Sigma) \rightarrow X(\Sigma)$ is an isomorphism of character varieties, and the maps $X_{SL_2(\mathbb{C})}(W_i) \rightarrow X(W_i)$ and $X_{SL_2(\mathbb{C})}(F) \rightarrow X(F)$ are local diffeomorphisms in a neighborhood of an isolated intersection point $\chi \in X_{SL_2(\mathbb{C})}^*(W_1) \cap X_{SL_2(\mathbb{C})}^*(W_2)$. Thus in fact $\lambda_{PSL_2(\mathbb{C})}(\Sigma) = \lambda_{SL_2(\mathbb{C})}(\Sigma)$.

- (v) This follows immediately from Definition 2.1.
- (vi) Any representation $\rho: \pi_1(\Sigma_1 \# \Sigma_2) \rightarrow PSL_2(\mathbb{C})$ is in fact an ordered pair of representations (ρ_1, ρ_2) , where $\rho_i: \pi_1(\Sigma_i) \rightarrow PSL_2(\mathbb{C})$. Moreover, if $g \in PSL_2(\mathbb{C})$, then $(g \cdot \rho_1, \rho_2)$ is not conjugate to (ρ_1, ρ_2) in general. In particular, if ρ_1 and ρ_2 are both nontrivial, we may use this fact to generate a curve of characters in $X(\Sigma_1 \# \Sigma_2)$ containing ρ . Thus, the isolated characters in $X^*(\Sigma_1 \# \Sigma_2)$ are characters of representations $\rho = (\rho_{triv}, \rho_2)$ with ρ_2 an irreducible representation of $\pi_1(\Sigma_2)$ and characters of representations $\rho = (\rho_1, \rho_{triv})$ with ρ_1 an irreducible representation of $\pi_1(\Sigma_1)$, where ρ_{triv} is the trivial representation. Hence

$$\lambda_{PSL_2(\mathbb{C})}(\Sigma_1 \# \Sigma_2) = \lambda_{PSL_2(\mathbb{C})}(\Sigma_1) + \lambda_{PSL_2(\mathbb{C})}(\Sigma_2).$$

□

We remark that this connected sum formula is more elegant than the corresponding formula for the $SL_2(\mathbb{C})$ -invariant. In [2] it is shown that

$$\lambda_{SL_2(\mathbb{C})}(\Sigma_1 \# \Sigma_2) = |H^1(\Sigma_2; \mathbb{Z}_2)| \lambda_{SL_2(\mathbb{C})}(\Sigma_1) + |H^1(\Sigma_1; \mathbb{Z}_2)| \lambda_{SL_2(\mathbb{C})}(\Sigma_2).$$

In contrast, the $PSL_2(\mathbb{C})$ -invariant is additive.

4 Computations

We now study a number of examples which will help clarify the relationship between the $PSL_2(\mathbb{C})$ - and $SL_2(\mathbb{C})$ -invariants.

In [2], the invariant $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ is computed for Σ a Seifert fibered integral homology sphere and for Σ the result of p/q -Dehn surgery on a twist knot as long as p/q is not a strict boundary slope for K_ξ , where K_ξ for $\xi \geq 1$ denotes the ξ -twist knot in S^3 depicted in Figure 1 below. Moreover, the strict boundary slopes for K_ξ are known and are all even. If Σ is an integral homology sphere, then $\lambda_{PSL_2(\mathbb{C})}(\Sigma) = \lambda_{SL_2(\mathbb{C})}(\Sigma)$ by Theorem 3.1 (iv). Thus we have the following:

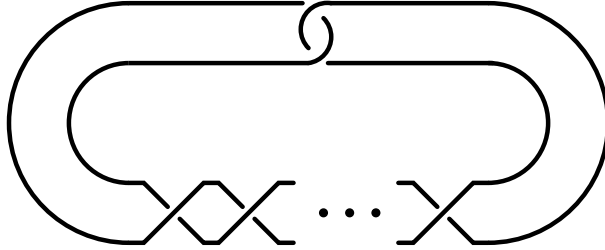


Figure 1 The twist knot K_ξ with a clasp and ξ half twists.

Theorem 4.1 (i) Suppose a_1, \dots, a_n are positive integers that are pairwise relatively prime, and denote by $\Sigma(a_1, \dots, a_n)$ the associated Seifert fibered homology sphere. Then

$$\lambda_{PSL_2(\mathbb{C})}(\Sigma(a_1, \dots, a_n)) = \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \frac{(a_{i_1} - 1)(a_{i_2} - 1)(a_{i_3} - 1)}{4}.$$

(ii) If Σ is the result of p/q -Dehn surgery on K_ξ , where ξ is even and p is odd, then

$$\lambda_{PSL_2(\mathbb{C})}(\Sigma) = \frac{1}{4}(\xi|4q + p| + (\xi - 2)|p| + 2|2\xi q - p| - 2\xi).$$

(iii) If Σ is the result of p/q -Dehn surgery on K_ξ , where ξ is odd and p is odd, then

$$\lambda_{PSL_2(\mathbb{C})}(K_\xi(p/q)) = \frac{1}{4}((\xi - 1)(|4q - p| + |p|) + 2|2\xi q + 4q - p| - 2\xi).$$

We now turn our attention to certain Seifert fibered spaces which are not integral homology spheres. For Seifert fibered spaces which are not \mathbb{Z}_2 -homology spheres, the $PSL_2(\mathbb{C})$ - and $SL_2(\mathbb{C})$ -invariants may differ. Let Σ be a non-Haken Seifert manifold whose base orbifold is of the form $S^2(p, q, r)$ where $p, q, r \geq 2$.

In [1], Proposition D, the authors compute the number of $PSL_2(\mathbb{C})$ characters of $\pi_1 \Sigma$. They note further that for all irreducible representations $\rho : \pi_1 \Sigma \rightarrow PSL_2(\mathbb{C})$ the cohomology group $H^1(\Sigma; \text{Ad} \rho)$ is zero. But arguing as in the proof of Theorem 2.1 of [2], we find that the character of any such representation in $X^*(W_1) \cap X^*(W_2)$ is an isolated character with multiplicity 1 if it is not the character of a representation which conjugates into D , with multiplicity 1/2 if it is the character of a representation which conjugates into $D - Z$, and with multiplicity 1/4 if it is the character of a representation which conjugates into Z . Thus, in this setting, counting the characters of $\pi_1(\Sigma)$ enables us to compute $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$.

Theorem 4.2 Let Σ be a non-Haken Seifert manifold whose base orbifold is of the form $S^2(p, q, r)$ where $p, q, r \geq 2$. Let $[\cdot]$ denote the greatest integer function. Then

$$\begin{aligned} \lambda_{PSL_2(\mathbb{C})}(\Sigma) = & \frac{1}{4}(p-1)(q-1)(r-1) + \left\lfloor \frac{\gcd(p, q)}{2} \right\rfloor + \left\lfloor \frac{\gcd(p, r)}{2} \right\rfloor + \left\lfloor \frac{\gcd(q, r)}{2} \right\rfloor \\ & - \left\lfloor \frac{\gcd(pq, pr, qr)}{2} \right\rfloor - \sigma(p, q, r), \end{aligned}$$

where $\sigma(p, q, r)$ is 1 if p, q , and r are all even and 0 otherwise.

Proof To see this, we recall Proposition D of [1]. It is well-known that $\pi_1(\Sigma)$ has a presentation of the form

$$\pi_1(\Sigma) = \langle x, y, h \mid h \text{ is central, } x^p = h^a, y^q = h^b, (xy)^r = h^c \rangle,$$

where $\gcd(a, p) = \gcd(b, q) = \gcd(c, r) = 1$. (See [9] for example.)

Theorem 4.3 (Ben Abdelghani, Boyer, Theorem D, [1])

(i) The number of reducible $PSL_2(\mathbb{C})$ -characters of $\pi_1(\Sigma)$ is

$$\left\lfloor \frac{|aqr + bpr + cpq|}{2} \right\rfloor + \begin{cases} 1 & \text{if } \gcd(p, q, r) \text{ is odd} \\ 2 & \text{if } \gcd(p, q, r) \text{ is even.} \end{cases} \quad (4.1)$$

(ii) The number of $PSL_2(\mathbb{C})$ -characters of $\pi_1(\Sigma)$ with image a dihedral group of order at least 4 is

$$\sigma(q, r) \left\lfloor \frac{p}{2} \right\rfloor + \sigma(p, r) \left\lfloor \frac{q}{2} \right\rfloor + \sigma(p, q) \left\lfloor \frac{r}{2} \right\rfloor - 2\sigma(p, q, r) \quad (4.2)$$

where $\sigma(m, n)$ is one if m and n are both even and zero otherwise.

(iii) The number of $PSL_2(\mathbb{C})$ -characters of $\pi_1(\Sigma)$ is

$$\begin{aligned} & \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor + \left\lfloor \frac{|aqr + bpr + cpq|}{2} \right\rfloor \\ & - \left\lfloor \frac{\gcd(pq, pr, qr)}{2} \right\rfloor + \left\lfloor \frac{\gcd(p, q)}{2} \right\rfloor + \left\lfloor \frac{\gcd(p, r)}{2} \right\rfloor + \left\lfloor \frac{\gcd(q, r)}{2} \right\rfloor + 1 \end{aligned} \quad (4.3)$$

Now the dihedral representations are precisely those which conjugate into D . Such a representation ρ contributes $1/2$ to $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$ if it conjugates into $D - Z$ and contributes $1/4$ to $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$ if it conjugates into Z . The reducible characters contribute nothing to $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$, and the remaining characters contribute 1 to $\lambda_{PSL_2(\mathbb{C})}(\Sigma)$.

Note that

$$H_1(\Sigma) \cong \mathbb{Z}/m_1 \oplus \mathbb{Z}/m_2,$$

where $m_1 = \gcd(p, q, r)$ and $m_2 = |aqr + bpr + cpq|/m_1$. Then $\pi_1(\Sigma)$ has exactly one character corresponding to a representation which conjugates into Z if p, q , and r are all even and has no such characters if at least one of p, q , and r is odd. It follows that

$$\lambda_{PSL_2(\mathbb{C})}(\Sigma) = (4.3) - (4.1) - (4.2)/2 - \sigma(p, q, r)/4.$$

Simplifying this expression, we obtain

$$\begin{aligned} \lambda_{PSL_2(\mathbb{C})}(\Sigma) = & \frac{1}{4}(p-1)(q-1)(r-1) + \left\lfloor \frac{\gcd(p, q)}{2} \right\rfloor + \left\lfloor \frac{\gcd(p, r)}{2} \right\rfloor + \\ & \left\lfloor \frac{\gcd(q, r)}{2} \right\rfloor - \left\lfloor \frac{\gcd(pq, pr, qr)}{2} \right\rfloor - \sigma(p, q, r), \end{aligned}$$

as desired. \square

We would like to determine $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ for comparison. As in the $PSL_2(\mathbb{C})$ -character variety, every character in $X_{SL_2(\mathbb{C})}^*(W_1) \cap X_{SL_2(\mathbb{C})}^*(W_2)$ is an isolated character, and for any such character χ_ρ the cohomology group $H^1(\Sigma; Ad\rho)$ is zero. Since $PSL_2(\mathbb{C})$ acts freely on the orbit of every irreducible representation, each such character has multiplicity 1 in the intersection cycle $X_{SL_2(\mathbb{C})}^*(W_1) \cdot X_{SL_2(\mathbb{C})}^*(W_2)$. Thus, $\lambda_{SL_2(\mathbb{C})}(\Sigma)$ is a precise count of the irreducible characters of Σ . We show

Theorem 4.4 *Let Σ be a non-Haken Seifert manifold whose base orbifold is of the form $S^2(p, q, r)$ where $p, q, r \geq 2$. Write $p = 2^\alpha p'$, $q = 2^\beta q'$, and $r = 2^\gamma r'$,*

where p' , q' , and r' are odd and α , β , and γ are greater than or equal to 0. Assume (by reordering p , q , and r as necessary) that $\alpha \geq \beta \geq \gamma$. Then

$$\lambda_{SL_2(\mathbb{C})}(\Sigma) = \frac{1}{4}(p-1)(q-1)(r-1) + \frac{\sigma(p,q)}{4}(r-1) + \frac{\sigma(p,r)}{4}(q-1) + \frac{\sigma(q,r)}{4}(p-1) + \xi_1 \left\lfloor \frac{\gcd(p,q)}{2} \right\rfloor + \xi_2 \left\lfloor \frac{\gcd(p,r)}{2} \right\rfloor + \xi_2 \left\lfloor \frac{\gcd(q,r)}{2} \right\rfloor - \xi_3 \left\lfloor \frac{\gcd(pq,pr,qr)}{2} \right\rfloor - 4\sigma(p,q,r),$$

$$\text{where } \begin{cases} \xi_1 = 2 & \text{if } \beta > 0, \\ \xi_1 = 1 & \text{otherwise;} \\ \xi_2 = 2 & \text{if } \gamma > 0 \text{ or if } \gamma = 0 \text{ and } \alpha = \beta > 0, \\ \xi_2 = 1 & \text{otherwise;} \\ \xi_3 = 2 & \text{if } \alpha = \beta > \gamma \text{ or if } \alpha > \beta = \gamma, \\ \xi_3 = 1 & \text{otherwise.} \end{cases}$$

In fact, one can check that this equality holds whenever at most one of p , q , and r is even, regardless of the parities of a , b , and c .

Proof Note that if $|aqr + bpr + cpq|$ is odd, then Σ is an \mathbb{Z}_2 -homology sphere, and $\lambda_{SL_2(\mathbb{C})}(\Sigma) = \lambda_{PSL_2(\mathbb{C})}(\Sigma)$ by Theorem 3.1 (iv). The assertion follows in this case since the formula of Theorem 4.2 agrees with the formula given here if $\beta = 0$. We are left with three cases:

Case 1: $\alpha = \beta = \gamma = 0$; a , b , and c even

Case 2: $\beta > 0$, $\gamma = 0$

Case 3: $\gamma > 0$

We prove Case 3. The remaining cases can be proved using similar arguments; we omit the proofs. Note that in Case 3, p , q , and r are even, so a , b , and c are odd.

We argue along the lines of [1]. Note that h is central in $\pi_1(\Sigma)$. It is easy to check that if $\rho(h) \neq \pm I$, then $\rho(\pi_1(\Sigma))$ is diagonalizable. Thus for any irreducible representation ρ we have $\rho(h) = \pm I$. We begin by counting the characters of representations with $\rho(h) = I$.

Since $\rho(h) = I$, the irreducible characters are in one-to-one correspondence with the irreducible characters of the triangle group

$$\Delta(p, q, r) = \langle x, y | x^p = y^q = (xy)^r = 1 \rangle.$$

Note that the character of any representation ρ of $\Delta(p, q, r)$ with $\rho(x) = \pm I$, $\rho(y) = \pm I$, or $\rho(xy) = \pm I$ is reducible. Hence the irreducible characters of $\Delta(p, q, r)$ all lie on the curves $C(j, k)$, $1 \leq j \leq (p/2) - 1$, $1 \leq k \leq (q/2) - 1$, in the character variety $X_{SL_2(\mathbb{C})}(\mathbb{Z}_p * \mathbb{Z}_q)$, given as follows: a character $\chi_z \in C(j, k)$ is the character of a representation ρ_z with

$$\rho_z(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } \rho_z(y) = \begin{pmatrix} z & 1 \\ z(\tau - z) - 1 & \tau - z \end{pmatrix}.$$

Here, $\lambda = e^{2\pi i j/p}$, $\mu = e^{2\pi i k/q}$, and $\tau = \mu + \mu^{-1}$.

Each such curve is isomorphic to \mathbb{C} via the map $z \rightarrow \chi_z$. Moreover the traces $\text{tr}(\rho_z(x))$ and $\text{tr}(\rho_z(y))$ are constant along the curves, and each curve corresponds to an allowable choice of $\text{tr}(\rho(x))$ and $\text{tr}(\rho(y))$ for $\rho \in R_{SL_2(\mathbb{C})}(\Delta(p, q, r))$.

Now $\text{tr}(\rho_z(xy)) = z(\lambda - \lambda^{-1}) + \lambda^{-1}\tau$. Since $\lambda \neq \lambda^{-1}$, all possible values of $\text{tr}(\rho_z(xy))$ are attained exactly once on the curve - i.e. the map $z \rightarrow \text{tr}(\rho_z(xy))$ is an isomorphism $\mathbb{C} \rightarrow \mathbb{C}$.

Note that any irreducible representation ρ of $\Delta(p, q, r)$ must satisfy $\rho(xy)^r = I$, so $\mathrm{tr}(\rho(xy)) = \nu + \nu^{-1}$ for $\nu = e^{2\pi il/r}$ for some l with $1 \leq l \leq (r/2) - 1$. Hence there are precisely $(r/2) - 1$ characters on each curve $C(j, k)$ which are characters in $X_{SL_2(\mathbb{C})}(\Delta(p, q, r))$. Summing over all curves $C(j, k)$, we find that we have counted $((p/2) - 1)((q/2) - 1)((r/2) - 1)$ distinct characters.

Unfortunately some of these characters may be reducible. To count the reducible characters of $\Delta(p, q, r)$ lying on the curves $C(j, k)$, $1 \leq j \leq (p/2) - 1$, $1 \leq k \leq (q/2) - 1$, we count the total number of reducible characters in $X_{SL_2(\mathbb{C})}(\Delta(p, q, r))$ and subtract the number of reducible characters χ_ρ corresponding to representations ρ with $\rho(x) = \pm I$, $\rho(y) = \pm I$, or $\rho(xy) = \pm I$.

Note that

$$H_1(\Delta(p, q, r)) \cong \mathbb{Z}/m_1 \oplus \mathbb{Z}/m'_2,$$

where $m_1 = \gcd(p, q, r)$ and $m'_2 = \gcd(pq, pr, qr)/m_1$. Then the number of reducible characters of $\Delta(p, q, r)$ is $2 + \gcd(pq, pr, qr)/2$.

If ρ is a reducible character in $X_{SL_2(\mathbb{C})}(\Delta(p, q, r))$ with $\rho(x) = I$, we see that $\rho(y) = \rho(xy)$. Since $\rho(y)^q = \rho(xy)^r = I$, we see that there are $(\gcd(q, r)/2) + 1$ such characters. Similarly there are $(\gcd(q, r)/2) + 1$ characters χ_ρ corresponding to representations ρ with $\rho(x) = -I$, $(\gcd(p, r)/2) - 1$ characters χ_ρ of representations ρ with $\rho(y) = I$ and $\rho(x) \neq \pm I$, and $(\gcd(p, r)/2) - 1$ characters χ_ρ of representations ρ with $\rho(y) = -I$ and $\rho(x) \neq \pm I$. Finally, there are $(\gcd(p, q)/2) - 1$ characters χ_ρ of representations ρ with $\rho(xy) = I$ and $\rho(x) = \rho(y)^{-1} \neq \pm I$ and $(\gcd(p, q)/2) - 1$ characters χ_ρ of representations ρ with $\rho(xy) = -I$ and $\rho(x) = \rho(y)^{-1} \neq \pm I$.

Thus, there are $\gcd(p, q) + \gcd(p, r) + \gcd(q, r) - 2$ reducible characters with $\rho(x) = \pm I$, $\rho(y) = \pm I$, or $\rho(xy) = \pm I$. It follows that there are

$$4 + (\gcd(pq, pr, qr)/2) - \gcd(p, q) - \gcd(p, r) - \gcd(q, r)$$

reducible characters of Σ lying on the curves $C(j, k)$, $1 \leq j \leq (p/2) - 1$, $1 \leq k \leq (q/2) - 1$. Hence there are

$$((p/2) - 1)((q/2) - 1)((r/2) - 1) - (\gcd(pq, pr, qr)/2) + \gcd(p, q) + \gcd(p, r) + \gcd(q, r) - 4$$

irreducible characters χ in $X_{SL_2(\mathbb{C})}(\Sigma)$ such that $\chi(h) = 2$.

It remains to count the irreducible characters χ in $X_{SL_2(\mathbb{C})}(\Sigma)$ with $\chi(h) = -2$. Note that a, b , and c are odd since $\gcd(p, a) = \gcd(q, b) = \gcd(r, c) = 1$. Then such characters are in one-to-one correspondence with the characters χ_ρ of the triangle group $\Delta(2p, 2q, 2r)$ corresponding to representations ρ which satisfy

$$\rho(x)^p = \rho(y)^q = \rho(xy)^r = -I.$$

These characters lie on the curves $C(j, k) = \{\rho_z | z \in \mathbb{C}\}$, $1 \leq j \leq p/2$, $1 \leq k \leq q/2$ in $X_{SL_2(\mathbb{C})}(\mathbb{Z}_{2p} * \mathbb{Z}_{2q})$, where $\lambda = e^{\pi i(2j-1)/p}$, $\mu = e^{\pi i(2k-1)/q}$, and ρ_z is defined as above. As above, it is easy to check that there are $r/2$ possible choices for $\mathrm{tr}(\rho(xy))$ on each curve $C(j, k)$, so there are $(p/2)(q/2)(r/2)$ characters of Σ on these curves.

Again, some of these characters may be reducible. One checks that there are $\gcd(pq, pr, qr)/2$ reducible characters on these curves if exactly two of $\mathrm{lcm}(p, q, r)/p$, $\mathrm{lcm}(p, q, r)/q$, and $\mathrm{lcm}(p, q, r)/r$ are odd, so either $\alpha = \beta \neq \gamma$ or $\alpha \neq \beta = \gamma$, and none otherwise. It follows that there are $(p/2)(q/2)(r/2) - \delta \gcd(pq, pr, qr)/2$ irreducible characters χ of Σ such that $\chi(h) = -2$, where $\delta = 1$ if exactly two of $\mathrm{lcm}(p, q, r)/p$, $\mathrm{lcm}(p, q, r)/q$, and $\mathrm{lcm}(p, q, r)/r$ are odd, so either $\alpha = \beta \neq \gamma$ or $\alpha \neq \beta = \gamma$, and $\delta = 0$ otherwise. Adding this to the number of irreducible

characters χ of Σ with $\chi(h) = 2$, we find that the total number of irreducible characters is

$$(p/2)(q/2)(r/2) + ((p/2) - 1)((q/2) - 1)((r/2) - 1) - \frac{\xi_3}{2} \gcd(pq, pr, qr) \\ + \gcd(p, q) + \gcd(p, r) + \gcd(q, r) - 4.$$

Finally, note that

$$(p/2)(q/2)(r/2) + ((p/2) - 1)((q/2) - 1)((r/2) - 1) = \frac{1}{4}(p - 1)(q - 1)(r - 1) \\ + \frac{1}{4}(r - 1) + \frac{1}{4}(q - 1) + \frac{1}{4}(p - 1).$$

This proves Case 3, since each character in $X_{SL_2(\mathbb{C})}^*(\Sigma)$ contributes 1 to the $SL_2(\mathbb{C})$ -Casson invariant. \square

5 Reflections

We conclude the paper with some observations, questions, and conjectures. A detailed analysis of these points will be the topic of a future paper.

The following statements are either known or should be straightforward to prove using standard techniques. (See [8], [4], and [5] for similar arguments.)

1. A representation $\rho : \pi_1 \rightarrow PSL_2(\mathbb{C})$ lifts to a representation into $SL_2(\mathbb{C})$ if and only if $w_2(\rho) = 0 \in H^2(\Sigma; \mathbb{Z}_2)$.
2. If $w_2(\rho) = 0$, then there are exactly $|H^1(\Sigma; \mathbb{Z}_2)|$ distinct lifts of ρ . If one lift is $\tilde{\rho}$, then all lifts are given by $\alpha\tilde{\rho}$ for $\alpha \in H^1(\Sigma; \mathbb{Z}_2) = Hom(\pi_1(\Sigma), \mathbb{Z}_2)$, where $\alpha\tilde{\rho}(\gamma) = \alpha(\gamma)\tilde{\rho}(\gamma)$ for $\gamma \in \pi_1(\Sigma)$.
3. Lifts of conjugate representations are conjugate.
4. Distinct lifts of a single irreducible representation ρ with $w_2(\rho) = 0$ are not conjugate if the image of ρ does not conjugate into D . If the image of ρ conjugates into $D - Z$, then the lifts of ρ are conjugate in pairs: i.e. each lift $\tilde{\rho}$ of ρ is conjugate to exactly one other lift $\alpha\tilde{\rho}$ of ρ . If the image of ρ conjugates into Z , then each lift $\tilde{\rho}$ of ρ is conjugate to exactly three other lifts $\alpha\tilde{\rho}$ of ρ .
5. A representation $\rho : \pi_1 \rightarrow PSL_2(\mathbb{C})$ with $w_2(\rho) = 0$ is reducible if and only if its lifts are reducible.
6. A representation $\rho : \pi_1 \rightarrow PSL_2(\mathbb{C})$ with $w_2(\rho) = 0$ is isolated if and only if its lifts are isolated.

As long as all of these statements are correct, it is clear that the $PSL_2(\mathbb{C})$ -characters χ with $w_2(\chi) = 0$ should contribute $\frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|} \lambda_{SL_2(\mathbb{C})}(\Sigma)$ to the $PSL_2(\mathbb{C})$ invariant. Thus,

Conjecture 5.1 For any closed, orientable 3-manifold Σ ,

$$\lambda_{PSL_2(\mathbb{C})}(\Sigma) = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|} \lambda_{SL_2(\mathbb{C})}(\Sigma) + \sum_{\chi} n_{\chi},$$

where the sum is taken over all isolated characters $\chi \in X^*(W_1) \cap X^*(W_2)$ with $w_2(\chi) \neq 0$.

Note that a surgery formula for the $SL_2(\mathbb{C})$ -Casson invariant exists for certain surgeries. (See [6] and [7].) Thus, the piece of the $PSL_2(\mathbb{C})$ -Casson invariant counting characters with $w_2 = 0$ can be easily computed in many cases.

It is clear from the above discussion that we can define invariants $\lambda_\alpha(\Sigma)$ counting the $PSL_2(\mathbb{C})$ -characters with $w_2 = \alpha$ for each α in $H^2(\Sigma; \mathbb{Z}_2)$. Then $\lambda_{PSL_2(\mathbb{C})}(\Sigma) = \sum_\alpha \lambda_\alpha(\Sigma)$. With this notation, Conjecture 4.1 is the assertion $\lambda_0 = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|} \lambda_{SL_2(\mathbb{C})}(\Sigma)$.

A similar study relating the $SO(3)$ -Casson invariant to the $SU(2)$ -Casson invariant was undertaken in [8]. In this setting, it was shown that for each element α of $H^2(\Sigma; \mathbb{Z}_2)$, the contribution $\lambda_{SO(3),\alpha}$ of conjugacy classes of $SO(3)$ -representations with $w_2 = \alpha$ to the $SO(3)$ -invariant was $\frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|} \lambda_{SU(2)}(\Sigma)$. Thus, in the $SO(3)$ -theory, we find that $\lambda_{SO(3),\alpha} = \lambda_{SO(3),0}$ for each $\alpha \in H^2(\Sigma; \mathbb{Z}_2)$. Since for closed 3-manifolds Σ we have $H^2(\Sigma; \mathbb{Z}_2) \cong H^1(\Sigma; \mathbb{Z}_2)$, we see that $\lambda_{SO(3)}(\Sigma) = \lambda_{SU(2)}(\Sigma)$.

Our investigation of Seifert fibered spaces shows that the relationship between the $PSL_2(\mathbb{C})$ - and $SL_2(\mathbb{C})$ -theories is fundamentally different from the relationship between the $SO(3)$ - and $SU(2)$ -theories. For example, let us consider a Seifert fibered space Σ with base orbifold $S^2(p, q, r)$, where $p = 4$, $q = 6$, and $r = 8$. Theorem 3.2 tells us that $\lambda_{PSL_2(\mathbb{C})}(\Sigma) = 101/4$. However by Theorem 3.3, we have $\lambda_{SL_2(\mathbb{C})}(\Sigma) = 30$, so $\lambda_0 = 30/4$. It follows that $\sum_{\alpha \neq 0} \lambda_\alpha = 71/4$, so clearly there exists an $\alpha \in H^2(\Sigma; \mathbb{Z}_2)$ such that $\lambda_\alpha(\Sigma) \neq \lambda_0(\Sigma)$. Equivalently, $\lambda_{PSL_2(\mathbb{C})}(\Sigma) \neq \lambda_{SL_2(\mathbb{C})}(\Sigma)$. This dichotomy between the compact and noncompact theories is most interesting. We are left with the following problem:

Problem 5.2 Find a surgery formula for $\lambda_\alpha(\Sigma)$ for each $\alpha \in H^2(\Sigma; \mathbb{Z}_2)$.

These issues will be addressed in a future paper.

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